Online Lecture Note

Introduction to Three-dimensional Incompressible CFD

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CONTENTS

1. Introduction

This lecture note introduces the basis of finite-difference method (FDM) for solving three-dimensional incompressible Navier-Stokes equations. Contents in this note are based on my lecture, ‘Computational Fluid Dynamics (CFD)’ at 6 semester of undergraduate school.

As in Fig.1, applied mathematician started CFD researches before 1950. The main stream in 1950 was to develop time-marching methods for solving compressible Euler equations. Unfortunately, such methods could not solve incompressible Navier-Stokes equations at that time. Because the continuity equation has no time derivative due to no density variation, the time-marching method is not applicable to the continuity equation. In 1965, Harlow and Welch [1] proposed a finite-difference method named Marker and Cell (MAC) method for solving incompressible Navier-Stokes equations. Until today, MAC-based methods are still in common use for simulating incompressible viscous flows. This note focuses on the methodology how MAC method solves three-dimensional incompressible Navier-Stokes equations.

First in this note, three-dimensional incompressible Navier-Stokes equations are derived from the compressible Navier-Stokes equations. Next, Marker and Cell (MAC) method applied to the equations is explained in detail. Further as numerical examples, some three-dimensional incompressible flows simulated by MAC-based method coupled with an immersed boundary method are introduced.

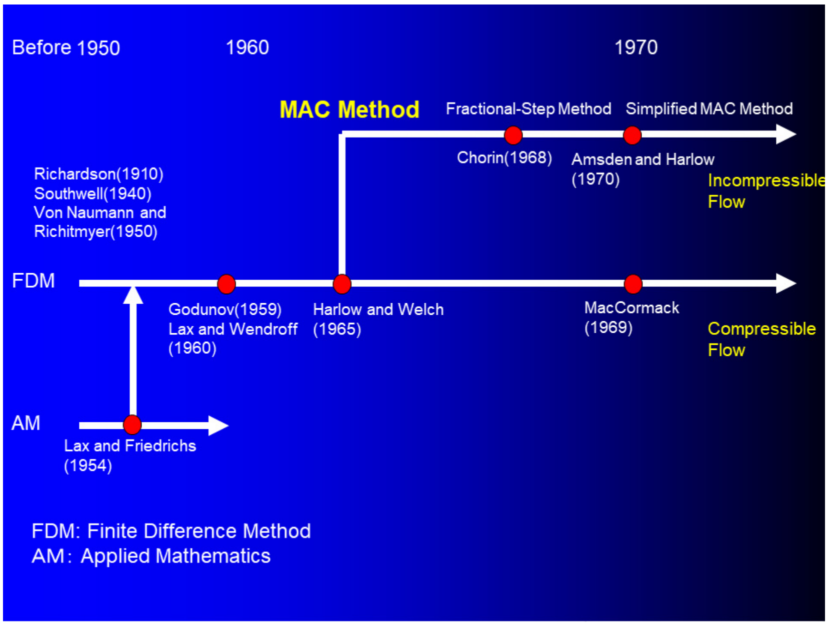


Fig. 1 Brief history of early CFD researches

2. Compressible Navier-Stokes Equations

Let me start from Compressible Navier-Stokes equations (CNS) using vector description as follows:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2-1}$$

$$(\rho \mathbf{u})_t + \nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla p = \nabla \cdot \Pi \tag{2-2}$$

$$e_t + \nabla \cdot (e + p)\mathbf{u} = \nabla \cdot (\Pi \cdot \mathbf{u}) + \nabla \cdot \mathbf{q} \tag{2-3}$$

where ρ, $\mathbf{u}$, p, Π, e and $\mathbf{q}$ are the density, velocity vector, pressure, viscous stress tensor, total internal energy per unit volume, and the vector of heat flux. t is the time and the subscription t means the partial derivation with respect to time. First, second, and third equations are respectively the mass conservation law, momentum conservation law, and the energy conservation law.

These equations can be also rewritten using tensor description:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \tag{2-4}$$

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j + \delta_{ij} p) = \frac{\partial}{\partial x_i}(\tau_{ij}) \tag{2-5}$$

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x_i}[(e + p)u_i] = \frac{\partial}{\partial x_i}\left(\tau_{ki} u_k + \kappa \frac{\partial T}{\partial x_i}\right) \tag{2-6}$$

where $(x_1, x_2, x_3) = (x, y, z)$ and $(u_1, u_2, u_3) = (u, v, w)$ for three dimensions in space. For example, originally $\frac{\partial}{\partial x_i}(\rho u_i) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho u_i)$. T and κ are the temperature and heat conductivity coefficient. τ_{ij} is the viscous stress tensor defined by

$$\tau_{ij} = \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] \quad (i, j = 1,2,3) \tag{2-7}$$

where μ and δ_{ij} are the molecular viscosity and Kronecker's delta. Second equation Eq. (2-5) is composed of three momentum equations along x, y, z in three dimensions.

CNS is not a closed system itself because the pressure p is still unknown. Assuming ideal gas, CNS can be closed by the equation of state for ideal gas:

$$p = \rho RT = (\gamma - 1)(e - \rho \mathbf{u}\mathbf{u}/2) = (\gamma - 1)(e - \rho u_i u_i/2) \tag{2-8}$$

where R and γ are the specific gas constant and specific heat ratio ($\gamma = 1.4$).

Three-dimensional CNS is written by the following tensor form:

$$\frac{\partial Q}{\partial t} + \frac{\partial F_i}{\partial x_i} = \frac{\partial F_{vi}}{\partial x_i} \quad (i = 1,2,3) \tag{2-9}$$

$$Q = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ e \end{bmatrix}, \quad F_i = \begin{bmatrix} \rho u_i \\ \rho u_1 u_i + \delta_{1i} p \\ \rho u_2 u_i + \delta_{2i} p \\ \rho u_3 u_i + \delta_{3i} p \\ (e + p)u_i \end{bmatrix}, \quad F_{vi} = \begin{bmatrix} 0 \\ \tau_{1i} \\ \tau_{2i} \\ \tau_{3i} \\ \tau_{ki} u_k + \kappa \partial T/\partial x_i \end{bmatrix}$$

where Q, F_i and F_{vi} are the vectors of unknown variables, convection and pressure terms (convection flux), and the diffusion terms (diffusion flux).

3. Incompressible Navier-Stokes Equations

Now let me derive three-dimensional incompressible Navier-Stokes equations from Eq. (2-9). First, as the most primary feature for incompressible flows, density is not varied: $\rho = \text{constant}$ (Note that exactly density is varied even if it is liquid). In addition, the internal energy e is not varied because of no compressibility. From these assumptions, Eq. (2-9) results in the following set of equations:

$$\frac{\partial Q}{\partial t} + \frac{\partial F_i}{\partial x_i} = \frac{\partial F_{vi}}{\partial x_i} \quad (i = 1,2,3) \tag{3-1}$$

$$Q = \begin{bmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad F_i = \begin{bmatrix} u_i \\ u_1 u_i + \delta_{1i}\, p/\rho \\ u_2 u_i + \delta_{2i}\, p/\rho \\ u_3 u_i + \delta_{3i}\, p/\rho \end{bmatrix}, \quad F_{vi} = \frac{1}{\rho}\begin{bmatrix} 0 \\ \tau_{1i} \\ \tau_{1i} \\ \tau_{1i} \end{bmatrix}$$

Next, let me nondimensionalize Eq. (3-1) with the following non-dimensional variables:

$$\bar{x}_j = \frac{x_j}{L},\ \bar{t} = \frac{t}{t_{ref}},\ \bar{u}_j = \frac{u_j}{V_\infty},\ \bar{p} = \frac{p}{\rho V_\infty^2},\ \bar{\mu} = \frac{\mu}{\mu_\infty} \tag{3-2}$$

The upper bar indicates the non-dimensional variable. L [m] and V_∞ [m/s] are the reference values of length and velocity. t_{ref} is a reference time derived as $t_{ref} = L/V_\infty$ and μ_∞ is the reference value of molecular viscosity coefficient. First equation of Eq. (3-1) corresponds to the continuity equation. The time-derivative term was lost because of no density variation. The continuity equation is nondimensionalized by the following manner:

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial}{\partial(\bar{x}_i L)}(\bar{u}_i V_\infty) = 0$$

$$\frac{\partial \bar{u}_i}{\partial \bar{x}_i} = 0 \tag{3-3}$$

Second to fourth equations correspond to the momentum equations. These three equations can be written by the same equation in a tensor form and the equation is nondimensionalized as follows:

$$\frac{\partial}{\partial t}(u_j) + \frac{\partial}{\partial x_i}(u_i u_j + \delta_{ij}\, p/\rho) = \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_i}$$

$$\frac{\partial}{\partial(\bar{t} t_{ref})}(\bar{u}_j V_\infty) + \frac{\partial}{\partial(\bar{x}_i L)}\left[\bar{u}_i V_\infty \bar{u}_j V_\infty + \delta_{ij}\, \bar{p}\rho V_\infty^2/\rho\right] = \frac{1}{\rho}\frac{\partial}{\partial(\bar{x}_i L)}\left(\bar{\tau}_{ij}\frac{\mu_\infty V_\infty}{L}\right)$$

$$\frac{V_\infty^2}{L}\frac{\partial}{\partial \bar{t}}(\bar{u}_j) + \frac{V_\infty^2}{L}\frac{\partial}{\partial \bar{x}_i}(\bar{u}_i \bar{u}_j + \delta_{ij}\bar{p}) = \frac{\mu_\infty V_\infty}{\rho L^2}\frac{\partial}{\partial \bar{x}_i}(\bar{\tau}_{ij})$$

$$\frac{\partial}{\partial \bar{t}}(\bar{u}_j) + \frac{\partial}{\partial \bar{x}_i}(\bar{u}_i \bar{u}_j + \delta_{ij}\bar{p}) = \frac{\mu_\infty}{\rho V_\infty L}\frac{\partial}{\partial \bar{x}_i}(\bar{\tau}_{ij}) \tag{3-4}$$

Since the Reynolds number is defined by $Re = \rho V_\infty L/\mu_\infty$, Eq.(3-4) can be rewritten by

$$\frac{\partial}{\partial \bar{t}}(\bar{u}_j) + \frac{\partial}{\partial \bar{x}_i}(\bar{u}_i \bar{u}_j + \delta_{ij}\bar{p}) = \frac{1}{Re}\frac{\partial}{\partial \bar{x}_i}(\bar{\tau}_{ij}) \tag{3-5}$$

According to Eq. (3-3) and Eq. (3-5), the set of equations removing the upper bar is summarized as the following vector form:

$$\frac{\partial Q}{\partial t}+\frac{\partial F_i}{\partial x_i}=\frac{1}{Re}\frac{\partial F_{\mathrm{v}i}}{\partial x_i}\quad (i=1,2,3) \tag{3-6}$$

$$Q=\begin{bmatrix}0\\u_1\\u_2\\u_3\end{bmatrix},\quad F_i=\begin{bmatrix}u_i\\u_1u_i+\delta_{1i}p\\u_2u_i+\delta_{2i}p\\u_3u_i+\delta_{3i}p\end{bmatrix},\quad F_{vi}=\begin{bmatrix}0\\\tau_{1i}\\\tau_{2i}\\\tau_{3i}\end{bmatrix}$$

where the viscous stress tensor has the same form with that for CNS as

$$\tau_{ij}=\mu\left[\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]\quad (i,j=1,2,3) \tag{3-7}$$

Next let me simplify Eq. (3-7) assuming incompressible flows. First, the molecular viscosity μ has already been nondimensionalized and is to be constant, setting it to one. In Eq. (3-7), the second term at right hand side coincides with the continuity equation as

$$\frac{\partial u_k}{\partial x_k}=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0$$

Then, the partial derivative of F_{vi} with respect to x_i are simplified by the following manner:

$$\frac{\partial F_{\mathrm{v}i}}{\partial x_i}=\frac{\partial F_{\mathrm{v}1}}{\partial x_1}+\frac{\partial F_{\mathrm{v}2}}{\partial x_2}+\frac{\partial F_{\mathrm{v}3}}{\partial x_3}=\begin{bmatrix}0\\\frac{\partial \tau_{11}}{\partial x_1}+\frac{\partial \tau_{12}}{\partial x_2}+\frac{\partial \tau_{13}}{\partial x_3}\\\frac{\partial \tau_{21}}{\partial x_1}+\frac{\partial \tau_{22}}{\partial x_2}+\frac{\partial \tau_{23}}{\partial x_3}\\\frac{\partial \tau_{31}}{\partial x_1}+\frac{\partial \tau_{32}}{\partial x_2}+\frac{\partial \tau_{33}}{\partial x_3}\end{bmatrix}$$

$$\begin{aligned}\frac{\partial \tau_{i1}}{\partial x_1}+\frac{\partial \tau_{i2}}{\partial x_2}+\frac{\partial \tau_{i3}}{\partial x_3}&=\frac{\partial}{\partial x_1}\left(\frac{\partial u_i}{\partial x_1}+\frac{\partial u_1}{\partial x_i}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial u_i}{\partial x_2}+\frac{\partial u_2}{\partial x_i}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_3}+\frac{\partial u_3}{\partial x_i}\right)\\&=\frac{\partial}{\partial x_1}\left(\frac{\partial u_i}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial u_i}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_3}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\\&=\frac{\partial^2 u_i}{\partial x_j^2}\end{aligned}$$

Consequently, τ_{ij} results in a simple form:

$$\tau_{ij}=\frac{\partial u_i}{\partial x_j}\quad (i,j=1,2,3) \tag{3-8}$$

Substituting Eq. (3-8) into Eq. (3-6), we obtain a set of fundamental equations for incompressible viscous flows written in vector description as follows:

$$\nabla\cdot\mathbf{u}=0 \tag{3-9}$$

$$\mathbf{u}_t + \nabla \cdot \mathbf{uu} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} \tag{3-10}$$

We call the set of Eq. (3-9) and Eq. (3-10) incompressible Navier-Stokes equations, or only the momentum equations may be called incompressible Navier-Stokes equations. Next shows the expanded forms.

・Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3-11}$$

・Momentum equations

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} + \frac{\partial wu}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{3-12}$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial wv}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{3-13}$$

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial ww}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \tag{3-14}$$

Hereafter, we call the incompressible Navier-Stokes equations, INS.

4. Poisson's Equation of Pressure

CNS has the time derivative term in all equations. As a typical finite-difference method, a time-marching method is applied to them for the time integration, whereas the continuity equation Eq. (3-10) has no time derivative term. Until today, we have no methods to solve Eq. (3-10) directly. Hallow and Welch [1] proposed alternative method to solve Eq. (3-10) indirectly in a Poisson's equation of pressure. Let me introduce the Poisson's equation in this section. First, the following parameters composing of convection terms and viscous terms in Eqs. (3-12) to (3-14) are defined:

$$F_u = -\frac{\partial uu}{\partial x} - \frac{\partial vu}{\partial y} - \frac{\partial wu}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{4-1}$$

$$F_v = -\frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} - \frac{\partial wv}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{4-2}$$

$$F_w = -\frac{\partial uw}{\partial x} - \frac{\partial vw}{\partial y} - \frac{\partial ww}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \tag{4-3}$$

Using these parameters, the momentum equations Eqs. (3-12) to (3-14) are rewritten as

$$\frac{\partial u}{\partial t} = F_u - \frac{\partial p}{\partial x} \tag{4-4}$$

$$\frac{\partial v}{\partial t} = F_v - \frac{\partial p}{\partial y} \tag{4-5}$$

$$\frac{\partial w}{\partial t} = F_w - \frac{\partial p}{\partial z} \tag{4-6}$$

To derive Poisson's equation of pressure, Eqs. (4-4) to (4-6) are partially differentiated with respect to x, y, and z. Then, we obtain the following partial differential equations:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(F_u - \frac{\partial p}{\partial x}\right) \tag{4-7}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right) = \frac{\partial}{\partial y}\left(F_v - \frac{\partial p}{\partial y}\right) \tag{4-8}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial t}\right) = \frac{\partial}{\partial z}\left(F_w - \frac{\partial p}{\partial z}\right) \tag{4-9}$$

Eqs. (4-7) to (4-9) are further added together as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial t}\right) = \frac{\partial}{\partial x}\left(F_u - \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(F_v - \frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(F_w - \frac{\partial p}{\partial z}\right) \tag{4-10}$$

Changing the sequence of partial derivation, we obtain the following form:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial F_u}{\partial x} + \frac{\partial F_v}{\partial y} + \frac{\partial F_w}{\partial z} - \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 p}{\partial z^2} \tag{4-11}$$

Finally, Poisson's equation of pressure can be derived as follows:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{\partial D}{\partial t} + \frac{\partial F_u}{\partial x} + \frac{\partial F_v}{\partial y} + \frac{\partial F_w}{\partial z} \tag{4-12}$$

where

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{4-13}$$

Eq. (4-13) corresponds the divergence of velocities, resulting in the continuity equation if it equals zero.

5. Marker and Cell (MAC) Method

Harlow and Welch proposed a numerical method based on finite-difference method (FDM) named Marker and Cell (MAC) method. We call it MAC method. The primary points are first to solve the momentum equations Eqs. (3-12) to (3-14), and Poisson's equation of pressure Eq. (4-12) instead of the continuity equation Eq. (3-11). The second is to employ a special treatment of computational grid named Staggered Grid. Usually unknown variables ρ, u_i, and e of CNS are defined and computed simultaneously at the same grid point. The unknown variables of MAC method are u_i and p. Harlow and Welch addressed that pressure was numerically oscillated as high or low value at the neighbor like a checkerboard if all the unknown variables are defined at the same grid point. To suppress the numerical oscillation, they defined the unknown variables at different grid points.

Figure 5-1 defines the grid points for unknown variables u, v, w, and p. Original grid points are numbered using the numbers i, j, and k in x, y, and z directions. A three-dimensional cubic volume with the eight grid points is defined as a Cell. The Cell number is (i,j,k). $p_{i,j,k}$ is pressure defined at the center of Cell. $u_{i,j,k}$, $v_{i,j,k}$, and $w_{i,j,k}$ are the velocities u, v, and w defined at the center of Cell surface.

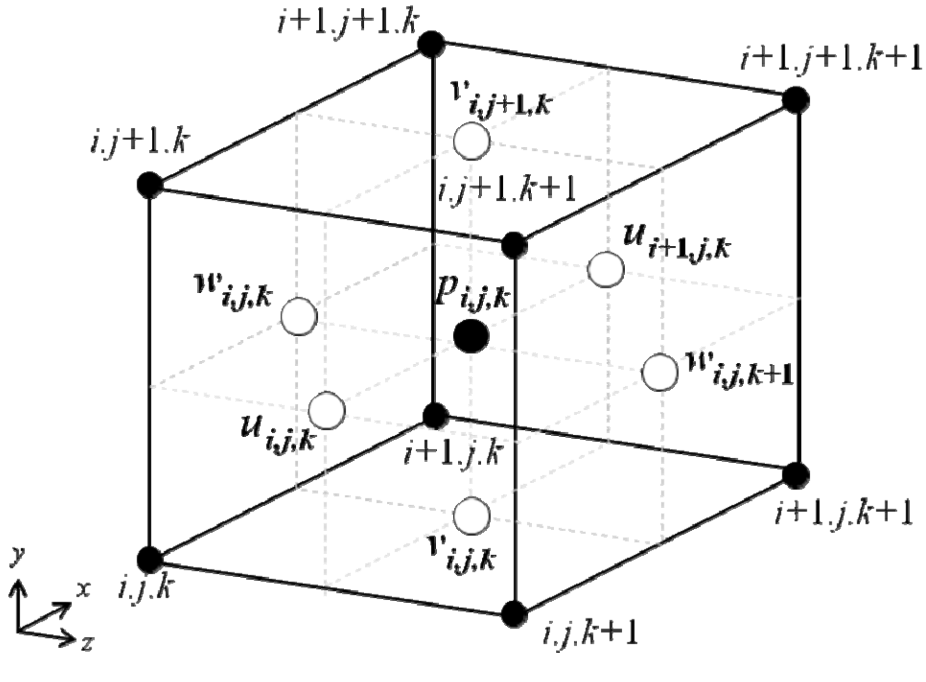


Fig. 5-1 Definition of grid points in Cell for staggered grid

According to the staggered grid, FDM is applied to the momentum equations and the Poisson equation of pressure. The finite-difference equations are solved at the grid points defined in Fig. 5-1. Hereafter, the notation $\left(\ \right)_{i,j,k}$ indicates the finite-difference value of term at each staggered grid point where the unknown variables $u_{i,j,k}$, $v_{i,j,k}$, $w_{i,j,k}$, and $p_{i,j,k}$ are defined. F_u, F_v, and F_w are differenced at (i,j,k) grid points where $u_{i,j,k}$, $v_{i,j,k}$, and $w_{i,j,k}$ are defined as follows:

$$\left(F_u\right)_{i,j,k} = -\left(\frac{\partial uu}{\partial x}\right)_{i,j,k} - \left(\frac{\partial vu}{\partial y}\right)_{i,j,k} - \left(\frac{\partial wu}{\partial z}\right)_{i,j,k} + \frac{1}{Re}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j,k} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j,k} + \left(\frac{\partial^2 u}{\partial z^2}\right)_{i,j,k}\right] \tag{5-1}$$

$$\left(F_v\right)_{i,j,k} = -\left(\frac{\partial uv}{\partial x}\right)_{i,j,k} - \left(\frac{\partial vv}{\partial y}\right)_{i,j,k} - \left(\frac{\partial wv}{\partial z}\right)_{i,j,k} + \frac{1}{Re}\left[\left(\frac{\partial^2 v}{\partial x^2}\right)_{i,j,k} + \left(\frac{\partial^2 v}{\partial y^2}\right)_{i,j,k} + \left(\frac{\partial^2 v}{\partial z^2}\right)_{i,j,k}\right] \tag{5-2}$$

$$\left(F_w\right)_{i,j,k} = -\left(\frac{\partial uw}{\partial x}\right)_{i,j,k} - \left(\frac{\partial vw}{\partial y}\right)_{i,j,k} - \left(\frac{\partial ww}{\partial z}\right)_{i,j,k} + \frac{1}{Re}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j,k} + \left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j,k} + \left(\frac{\partial^2 w}{\partial z^2}\right)_{i,j,k}\right] \tag{5-3}$$

To derive the finite-difference forms for terms at right hand side in Eqs. (5-1) to (5-3), unknown variables $u_{i+1/2,j,k}$, $v_{i,j+1/2,k}$, and $w_{i,j,k+1/2}$ defined at the intermediate grid point between those at $u_{i,j,k}$ and $u_{i+1,j,k}$; $v_{i,j,k}$ and $v_{i,j+1,k}$; $w_{i,j,k}$ and $w_{i,j,k+1}$ are further introduced as in Figs. 5.2, 5.3, and 5.3.

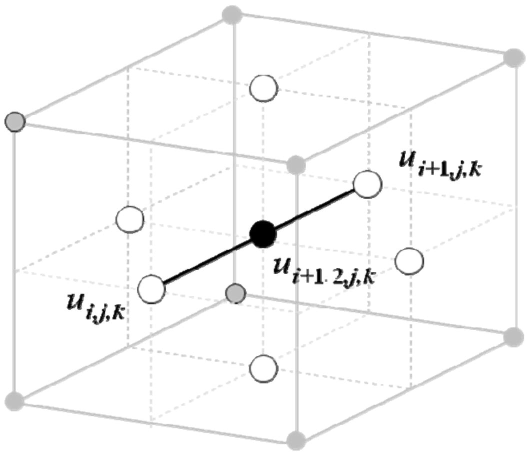


Fig. 5-2 Definition of $u_{i+1/2,j,k}$

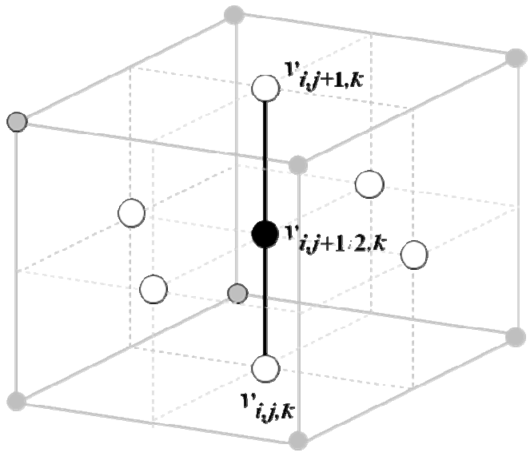


Fig. 5-3 Definition of $v_{i,j+1/2,k}$

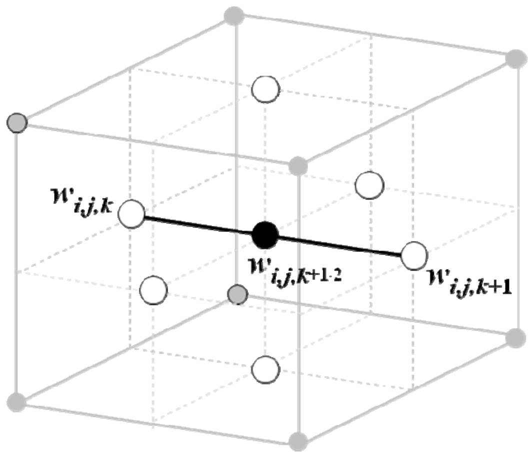


Fig. 5-4 Definition of $w_{i,j,k+1/2}$

The finite-difference forms for convection terms in Eqs. (5-1) to (5-3) are derived as follows:

$$\left(\frac{\partial uu}{\partial x}\right)_{i,j,k} = \frac{u_{i+1/2,j,k}u_{i+1/2,j,k} - u_{i-1/2,j,k}u_{i-1/2,j,k}}{\Delta x} = \frac{\left(u_{i+1,j,k} + u_{i,j,k}\right)\left(u_{i+1,j,k} + u_{i,j,k}\right) - \left(u_{i,j,k} + u_{i-1,j,k}\right)\left(u_{i,j,k} + u_{i-1,j,k}\right)}{4\Delta x} \tag{5-4}$$

$$\left(\frac{\partial vu}{\partial y}\right)_{i,j,k} = \frac{v_{i-1/2,j+1,k}u_{i,j+1/2,k} - v_{i-1/2,j,k}u_{i,j-1/2,k}}{\Delta y} = \frac{\left(v_{i-1,j+1,k} + v_{i,j+1,k}\right)\left(u_{i,j,k} + u_{i,j+1,k}\right) - \left(v_{i-1,j,k} + v_{i,j,k}\right)\left(u_{i,j-1,k} + u_{i,j,k}\right)}{4\Delta y} \tag{5-5}$$

$$\left(\frac{\partial wu}{\partial z}\right)_{i,j,k} = \frac{w_{i-1/2,j,k+1}u_{i,j,k+1/2} - w_{i-1/2,j,k}u_{i,j,k-1/2}}{\Delta z}$$
$$= \frac{\left(w_{i-1,j,k+1} + w_{i,j,k+1}\right)\left(u_{i,j,k} + u_{i,j,k+1}\right) - \left(w_{i-1,j,k} + w_{i,j,k}\right)\left(u_{i,j,k-1} + u_{i,j,k}\right)}{4\Delta z} \tag{5-6}$$

$$\left(\frac{\partial uv}{\partial x}\right)_{i,j,k} = \frac{u_{i+1,j-1/2,k}v_{i+1/2,j,k} - u_{i,j-1/2,k}v_{i-1/2,j,k}}{\Delta x}$$
$$= \frac{\left(u_{i+1,j-1,k} + u_{i+1,j,k}\right)\left(v_{i,j,k} + v_{i+1,j,k}\right) - \left(u_{i,j-1,k} + u_{i,j,k}\right)\left(v_{i-1,j,k} + v_{i,j,k}\right)}{4\Delta x} \tag{5-7}$$

$$\left(\frac{\partial vv}{\partial y}\right)_{i,j,k} = \frac{v_{i,j+1/2,k}v_{i,j+1/2,k} - v_{i,j-1/2,k}v_{i,j-1/2,k}}{\Delta y}$$
$$= \frac{\left(v_{i,j+1,k} + v_{i,j,k}\right)\left(v_{i,j,k} + v_{i,j+1,k}\right) - \left(v_{i,j-1,k} + v_{i,j,k}\right)\left(v_{i,j-1,k} + v_{i,j,k}\right)}{4\Delta y} \tag{5-8}$$

$$\left(\frac{\partial wv}{\partial z}\right)_{i,j,k} = \frac{w_{i,j-1/2,k+1}v_{i,j,k+1/2} - w_{i,j-1/2,k}v_{i,j,k-1/2}}{\Delta z}$$
$$= \frac{\left(w_{i,j-1,k+1} + w_{i,j,k+1}\right)\left(v_{i,j,k} + v_{i,j,k+1}\right) - \left(w_{i,j-1,k} + w_{i,j,k}\right)\left(v_{i,j,k-1} + v_{i,j,k}\right)}{4\Delta z} \tag{5-9}$$

$$\left(\frac{\partial uw}{\partial x}\right)_{i,j,k} = \frac{u_{i+1,j,k-1/2}w_{i+1/2,j,k} - u_{i,j,k-1/2}w_{i-1/2,j,k}}{\Delta x}$$
$$= \frac{\left(u_{i+1,j,k-1} + u_{i+1,j,k}\right)\left(w_{i,j,k} + w_{i+1,j,k}\right) - \left(u_{i,j,k-1} + u_{i,j,k}\right)\left(w_{i-1,j,k} + w_{i,j,k}\right)}{4\Delta x} \tag{5-10}$$

$$\left(\frac{\partial vw}{\partial y}\right)_{i,j,k} = \frac{v_{i,j+1,k-1/2}w_{i,j+1/2,k} - v_{i,j,k-1/2}w_{i,j-1/2,k}}{\Delta y}$$
$$= \frac{\left(v_{i,j+1,k-1} + v_{i,j+1,k}\right)\left(w_{i,j,k} + w_{i,j+1,k}\right) - \left(v_{i,j,k-1} + v_{i,j,k}\right)\left(w_{i,j-1,k} + w_{i,j,k}\right)}{4\Delta y} \tag{5-11}$$

$$\left(\frac{\partial ww}{\partial z}\right)_{i,j,k} = \frac{w_{i,j,k+1/2}w_{i,j,k+1/2} - w_{i,j,k-1/2}w_{i,j,k-1/2}}{\Delta z}$$
$$= \frac{\left(w_{i,j,k} + w_{i,j,k+1}\right)\left(w_{i,j,k} + w_{i,j,k+1}\right) - \left(w_{i,j,k-1} + w_{i,j,k}\right)\left(w_{i,j,k-1} + w_{i,j,k}\right)}{4\Delta z} \tag{5-12}$$

Numerical accuracy of Eqs. (5-3) to (5-12) is second-order in space.

Next the finite-difference forms of second-order space derivatives in Eqs. (5-1) to (5-3) are derived assuming second-order accuracy in space as follows:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j,k} = \frac{u_{i-1,j,k} - 2u_{i,j,k} + u_{i+1,j,k}}{(\Delta x)^2} \tag{5-13}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j,k} = \frac{u_{i,j-1,k} - 2u_{i,j,k} + u_{i,j+1,k}}{(\Delta y)^2} \tag{5-14}$$

$$\left(\frac{\partial^2 u}{\partial z^2}\right)_{i,j,k} = \frac{u_{i,j,k-1} - 2u_{i,j,k} + u_{i,j,k+1}}{(\Delta z)^2} \tag{5-15}$$

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{i,j,k} = \frac{v_{i-1,j,k} - 2v_{i,j,k} + v_{i+1,j,k}}{(\Delta x)^2} \tag{5-16}$$

$$\left(\frac{\partial^2 v}{\partial y^2}\right)_{i,j,k} = \frac{v_{i,j-1,k} - 2v_{i,j,k} + v_{i,j+1,k}}{(\Delta y)^2} \tag{5-17}$$

$$\left(\frac{\partial^2 v}{\partial z^2}\right)_{i,j,k} = \frac{v_{i,j,k-1} - 2v_{i,j,k} + v_{i,j,k+1}}{(\Delta z)^2} \tag{5-18}$$

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j,k} = \frac{w_{i-1,j,k} - 2w_{i,j,k} + w_{i+1,j,k}}{(\Delta x)^2} \tag{5-19}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j,k} = \frac{w_{i,j-1,k} - 2w_{i,j,k} + w_{i,j+1,k}}{(\Delta y)^2} \tag{5-20}$$

$$\left(\frac{\partial^2 w}{\partial z^2}\right)_{i,j,k} = \frac{w_{i,j,k-1} - 2w_{i,j,k} + w_{i,j,k+1}}{(\Delta z)^2} \tag{5-21}$$

Differencing all the terms in $(F_u)_{i,j,k}$, $(F_v)_{i,j,k}$, and $(F_w)_{i,j,k}$ was completed. The remaining terms are the time derivative and the pressure derivatives as in Eqs. (4-4) to (4-6). The pressure derivatives are differenced at (i,j,k) grid points where $u_{i,j,k}$, $v_{i,j,k}$, and $w_{i,j,k}$ are defined as follows:

$$\left(\frac{\partial p}{\partial x}\right)_{i,j,k} = \frac{p_{i,j,k} - p_{i-1,j,k}}{\Delta x} \tag{5-22}$$

$$\left(\frac{\partial p}{\partial y}\right)_{i,j,k} = \frac{p_{i,j,k} - p_{i,j-1,k}}{\Delta y} \tag{5-23}$$

$$\left(\frac{\partial p}{\partial z}\right)_{i,j,k} = \frac{p_{i,j,k} - p_{i,j,k-1}}{\Delta z} \tag{5-24}$$

The finite-difference forms of Eqs. (4-4) to (4-6) are written as

$$\left(\frac{\partial u}{\partial t}\right)_{i,j,k} = \left(F_u\right)_{i,j,k} - \left(\frac{\partial p}{\partial x}\right)_{i,j,k} \tag{5-25}$$

$$\left(\frac{\partial v}{\partial t}\right)_{i,j,k} = \left(F_v\right)_{i,j,k} - \left(\frac{\partial p}{\partial y}\right)_{i,j,k} \tag{5-26}$$

$$\left(\frac{\partial w}{\partial t}\right)_{i,j,k} = \left(F_w\right)_{i,j,k} - \left(\frac{\partial p}{\partial z}\right)_{i,j,k} \tag{5-27}$$

Time derivatives of Eqs. (5-25) to (5-27) are differenced with first-order accuracy as follows:

$$\frac{u^{n+1}_{i,j,k} - u^{n}_{i,j,k}}{\Delta t} = \left(F_u\right)_{i,j,k} - \left(\frac{\partial p}{\partial x}\right)_{i,j,k} \tag{5-28}$$

$$\frac{v^{n+1}_{i,j,k} - v^{n}_{i,j,k}}{\Delta t} = \left(F_v\right)_{i,j,k} - \left(\frac{\partial p}{\partial y}\right)_{i,j,k} \tag{5-29}$$

$$\frac{w^{n+1}_{i,j,k} - w^{n}_{i,j,k}}{\Delta t} = \left(F_w\right)_{i,j,k} - \left(\frac{\partial p}{\partial z}\right)_{i,j,k} \tag{5-30}$$

where the notation n at the superscript of unknown variables corresponds to the number of time iteration : n time-step. The variables at n time-step are known values and the unknown variables at $n+1$ time-step are obtained by the time-marching method as

$$u^{n+1}_{i,j,k} = u^{n}_{i,j,k} + \Delta t\left[\left(F_u\right)_{i,j,k} - \left(\frac{\partial p}{\partial x}\right)_{i,j,k}\right] \tag{5-31}$$

$$v^{n+1}_{i,j,k} = v^{n}_{i,j,k} + \Delta t\left[\left(F_v\right)_{i,j,k} - \left(\frac{\partial p}{\partial y}\right)_{i,j,k}\right] \tag{5-32}$$

$$v^{n+1}_{i,j,k} = v^{n}_{i,j,k} + \Delta t\left[\left(F_w\right)_{i,j,k} - \left(\frac{\partial p}{\partial z}\right)_{i,j,k}\right] \tag{5-33}$$

Eqs. (5-31) to (5-33) are the final finite-difference forms for the momentum equations of Eqs. (3-11) to (3-13).

Next the Poisson's equation of pressure in Eq.(4-12) is differenced at the grid point (i,j,k) where $p_{i,j,k}$ is defined as follows:

$$\left(\frac{\partial^2 p}{\partial x^2}\right)_{i,j,k}+\left(\frac{\partial^2 p}{\partial y^2}\right)_{i,j,k}+\left(\frac{\partial^2 p}{\partial z^2}\right)_{i,j,k}=\left(\frac{\partial D}{\partial t}\right)_{i,j,k}+\left(\frac{\partial F_u}{\partial x}\right)_{i,j,k}+\left(\frac{\partial F_v}{\partial y}\right)_{i,j,k}+\left(\frac{\partial F_w}{\partial z}\right)_{i,j,k} \tag{5-34}$$

The terms in left hand side are generally solved by a relaxation method and those in right hand side is solved algebraically. Attention to solve Eq. (5-34) is that the first term in right hand side is a time-derivative of D defined as Eq. (4-13) . If this term is differenced in first-order accuracy, then

$$\left(\frac{\partial D}{\partial t}\right)_{i,j,k}=\frac{D^{n+1}-D^n}{\Delta t} \tag{5-35}$$

Because unknown variable D^{n+1} at $n+1$ time-step is contained in Eq. (5-35), this variable and the terms at left hand side in Eq. (5-34) should be solved at the same time. However, the calculation may be almost impossible. Harlow and Welch decided that D^{n+1} is forced to be zero assuming the continuity equation is satisfied at $n+1$ time step as

$$D^{n+1}=0 \tag{5-36}$$

Then, Eq. (5-35) was simplified to

$$\left(\frac{\partial D}{\partial t}\right)_{i,j,k}=-\frac{D^n}{\Delta t} \tag{5-37}$$

Eq. (5-37) can be solved algebraically.

The second to fourth terms at right hand side in Eq. (5-34) are the space differences of F_u, F_v, and F_w. They are obtained by further differences on F_u, F_v, and F_w as follows:

$$\left(\frac{\partial F_u}{\partial x}\right)_{i,j,k}=\frac{(F_u)_{i+1.j.k}-(F_u)_{i.j.k}}{\Delta x} \tag{5-38}$$

$$\left(\frac{\partial F_v}{\partial y}\right)_{i,j,k}=\frac{(F_u)_{i.j+1.k}-(F_v)_{i.j.k}}{\Delta y} \tag{5-39}$$

$$\left(\frac{\partial F_w}{\partial z}\right)_{i,j,k}=\frac{(F_w)_{i.j.k+1}-(F_w)_{i.j.k}}{\Delta z} \tag{5-40}$$

Defining the set of terms at right hand side in Eq. (5.34) to $f_{i,j,k}$ as

$$f_{i,j,k}=\left(\frac{\partial D}{\partial t}\right)_{i,j,k}+\left(\frac{\partial F_u}{\partial x}\right)_{i,j,k}+\left(\frac{\partial F_v}{\partial y}\right)_{i,j,k}+\left(\frac{\partial F_w}{\partial z}\right)_{i,j,k} \tag{5-41}$$

Eq. (5-34) is rewritten by

$$\left(\frac{\partial^2 p}{\partial x^2}\right)_{i,j,k}+\left(\frac{\partial^2 p}{\partial y^2}\right)_{i,j,k}+\left(\frac{\partial^2 p}{\partial z^2}\right)_{i,j,k}=f_{i,j,k} \tag{5-42}$$

The terms in left hand side are differenced in second-order space accuracy as follows:

$$\frac{p_{i+1,j,k}-2p_{i,j,k}+p_{i-1,j,k}}{(\Delta x)^2}+\frac{p_{i,j+1,k}-2p_{i,j,k}+p_{i,j-1,k}}{(\Delta y)^2}+\frac{p_{i,j,k+1}-2p_{i,j,k}+p_{i,j,k-1}}{(\Delta z)^2}=f_{i,j,k} \tag{5-43}$$

If $p_{i,j,k}$ is derived from Eq. (5-43), then we obtain the following equation:

$$p_{i,j,k}=\frac{1}{L}\left[\frac{p_{i+1,j,k}+p_{i-1,j,k}}{(\Delta x)^2}+\frac{p_{i,j+1,k}+p_{i,j-1,k}}{(\Delta y)^2}+\frac{p_{i,j,k+1}+p_{i,j,k-1}}{(\Delta z)^2}-f_{i,j,k}\right] \tag{5-44}$$

where $L=2/(\Delta x)^2+2/(\Delta y)^2+2/(\Delta z)^2$. SOR method can be applied to Eq. (5-44) as

$$p^{n+1}_{i,j,k}=(1-\omega)p^{n}_{i,j,k}+\frac{\omega}{L}\left[\frac{p^{n}_{i+1,j,k}+p^{n+1}_{i-1,j,k}}{(\Delta x)^2}+\frac{p^{n}_{i,j+1,k}+p^{n+1}_{i,j-1,k}}{(\Delta y)^2}+\frac{p^{n}_{i,j,k+1}+p^{n+1}_{i,j,k-1}}{(\Delta z)^2}-f_{i,j,k}\right] \tag{5-45}$$

where ω is the over-relaxation parameter and it sets to a value in $1<\omega<2$. Finally, the finite-difference equations Eqs. (5-31) to (5-33) and Eq. (5-45) are solved simultaneously.

When incompressible flow problems are solved by MAC method, boundary conditions should be carefully treated.

Figure 5-5 shows the grid points near the solid wall boundary at $y=0$. The grid points $(i,1,k)$ and $(i+1,1,k)$ are located on the solid wall boundary. A boundary layer forms on the solid wall boundary in incompressible viscous flows. All the velocities u , v , and w on the solid wall boundary are to be zero. As the boundary layer approximation, pressure in the boundary layer is not changed toward normal direction from the solid wall boundary: $\partial p/\partial y=0$. Using staggered grid, only $v_{i,1,k}$ can be defined on the solid wall boundary at $y=0$ in Fig. 5-5, setting $v_{i,1,k}=0$. However, other variables are not defined on the boundary. To resolve this concern, a reflecting boundary condition is applied to $u_{i,1,k}$ and $w_{i,1,k}$ (not seen in Fig.4-5) to satisfy zero velocity at the solid wall boundary. For example, outer grid point $(i,0,k)$ is further added and $u_{i,0,k}$ is set to $u_{i,0,k}=-u_{i,1,k}$ Then, the boundary treatment satisfies $u=0$ on the boundary. Also $p_{i,0,k}$ is set to $p_{i,0,k}=p_{i,1,k}$ according to $\partial p/\partial y=0$.

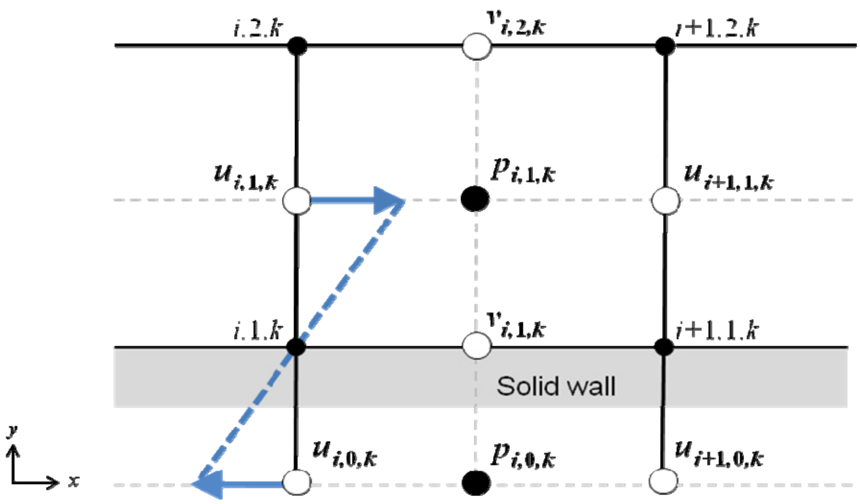


Fig. 5-5 Treatment of boundary conditions

6. Fractional Step Method and SMAC Method

Eqs. (5-31) to (5-33) and Eq. (5-45) are written again in the vector description as

$$\mathbf{u}_t + \nabla \cdot \mathbf{uu} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} \tag{6-1}$$

$$\nabla^2 p = -\nabla \cdot \left(\mathbf{u}_t + \nabla \cdot \mathbf{uu} - \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} \right) \tag{6-2}$$

MAC method solves Eq. (6-1) by the time-marching method and Eq. (6-2) by a relaxation method. As typical methods modifying MAC method, the fractional step method [2] and SMAC (Simplified MAC) method [3] are well known. In this section, these methods are further applied to Eqs. (6-1) and (6-2).

The finite difference forms of Eqs. (6-1) and (6-2) are obtained as follows:

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \left(\mathbf{F}^n - \nabla p^n \right) \tag{6-3}$$

$$\nabla^2 p^{n+1} = \nabla \cdot \mathbf{u}^n / \Delta t - \nabla \cdot \left(\nabla \cdot \mathbf{uu} - \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} \right)^n \tag{6-4}$$

The fractional step method divides Eq. (6-3) to two equations. Resultantly Eq. (6-4) can be simplified as

$$\mathbf{u}^* = \mathbf{u}^n + \Delta t \cdot \mathbf{F}^n \tag{6-5}$$

$$\nabla^2 p^{n+1} = \nabla \cdot \mathbf{u}^* / \Delta t \tag{6-6}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t \cdot \nabla p^{n+1} \tag{6-7}$$

$\mathbf{u}^*$ is the intermediate value in time between $\mathbf{u}^n$ and $\mathbf{u}^{n+1}$. The distinguished point for the fractional step method is the simplification of Poisson's equation of pressure. All the space derivatives of convection and viscous terms at right hand side in Eq. (6-4) are omitted.

SMAC method is further applied to Eqs. (6-5) to (6-7). Then, the following four-step equations are obtained:

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\left(\mathbf{F}^n - \nabla p^n\right) \tag{6-8}$$

$$\nabla^2 \phi^{n+1} = \nabla \cdot \mathbf{u}^* / \Delta t \tag{6-9}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t \cdot \nabla \phi^{n+1} \tag{6-10}$$

$$p^{n+1} = p^n + \phi^{n+1} \tag{6-11}$$

Eq. (6-8) forms rather the same equation with Eq. (6-3). A parameter ϕ is further introduced. ϕ^{n+1} is defined by $\phi^{n+1} = p^{n+1} - p^n$ as in Eq. (6-11) and Poisson's equation of ϕ is solved instead of that of pressure. Generally, the boundary conditions for pressure itself (Direchlet's boundary condition) or the derivative of pressure (Neumann's boundary condition) should be specified when Eq. (6-6) is solved, while all the boundary conditions can be specified to zero for both Direchlet's and Neumann's boundary conditions when Eq. (6-9) is solved, simplifying the boundary treatment for the Poisson's equation.

7. Immersed Boundary Method

Peskin [4] proposed the immersed boundary(IB) method for simulating blood flows in a heart. In the IB method, Cartesian grid is employed for the calculation of flow field. The geometry of the heart is immersed in the Cartesian grid and the effect of the immersed boundary is imposed to the fundamental equations as a source term. Clarke et al. [5] developed a similar method for inviscid flows using the Cartesian grid. Udaykumar et al. [6] extended this method to unsteady viscous flows. Mittal and Iaccarino [7] surveyed these methods and all of them were redefined as the IB method.

This lecture note introduces an IB method developed by us [8]. A primary advantage of this method is its simplicity as compared with the existing IB methods.

IB methods are categorized to two approaches. One is Feedback Forcing Method and another is Direct Forcing Method. IB method here is based on the Direct Forcing method.

First, external force **G** is added to Eqs. (4-4) to (4-6) for installing IB method into INS as

$$\frac{\partial u}{\partial t} = F_u - \frac{\partial p}{\partial x} + G_x \tag{7-1}$$

$$\frac{\partial v}{\partial t} = F_v - \frac{\partial p}{\partial y} + G_y \tag{7-2}$$

$$\frac{\partial w}{\partial t} = F_w - \frac{\partial p}{\partial z} + G_z \tag{7-3}$$

The time-marching method is applied to Eqs. (7-1) to (7-3) and the set of equations is written in the following vector form:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = \mathbf{F}^n - \nabla p^n + \mathbf{G}^n \tag{7-4}$$

The external force **G** is defined by

$$\mathbf{G}^n = -\mathbf{F}^n + \nabla p^n + \frac{\mathbf{u}^{IB} - \mathbf{u}^n}{\Delta t} \tag{7-5}$$

where $\mathbf{u}^{IB}$ is the corrected velocity vector obtained by IB method. This process means $\mathbf{u}^{n+1}$ is forcibly replaced by $\mathbf{u}^{IB}$. In this section, applying the IB method to SMAC method is main issue. Let me briefly explain how $\mathbf{u}^{IB}$ is calculated by our IB method. The IB method here employs a simple interpolation for obtaining $\mathbf{u}^{IB}$. Fig. 7-1 shows the velocities and pressure near the solid body. Red surface is a part of solid body which cuts the cubic composed of eight black points. The surface moves at u_s velocity. $u_{i,j,k}$ is calculated by the linear interpolation between $u_{i+1,j,k}$ and the velocity element of u_s in x direction and $u_{i,j,k}^{IB}$ is stored for calculating the external force G_x. $v_{i,j,k}^{IB}$ and $w_{i,j,k}^{IB}$ are obtained by the same manner.

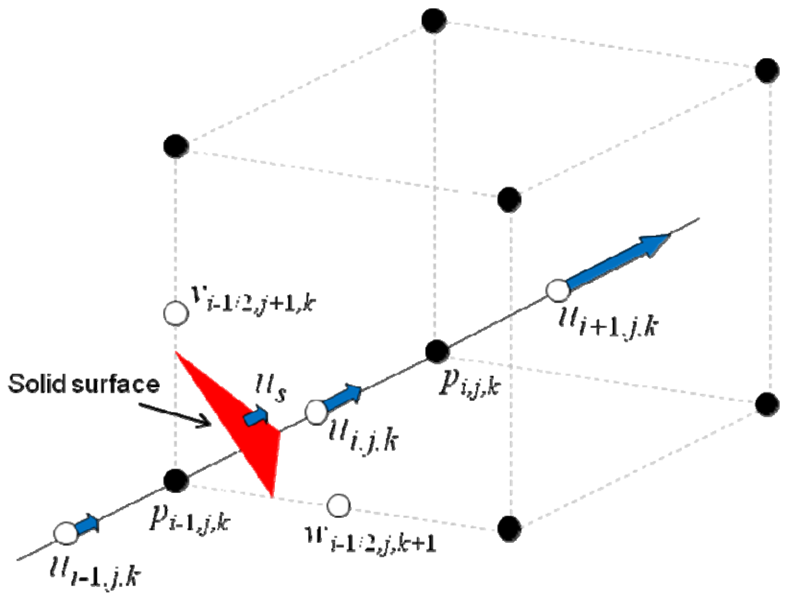


Fig. 7-1 Interpolation of $\mathbf{u}^{IB}$ along x-direction

8. Numerical Examples

We simulated incompressible viscous flows over a sphere using our IB method. Fig. 8.1 shows the computational grid for flow field and the surface mesh for the sphere. The surface mesh was immersed in the computational grid region. The computational grid region has 10D × 7.5D × 7.5 D volumes, where D is the diameter of sphere. A uniform flow was set at the inlet and the other boundaries were determined by the Sommerfeld's radiation condition. Grid dependency was checked by using different four computational grids as shown in Table 8.1.

Figure 8-2 plots the obtained vortex separation lengths after sphere compared with the existing numerical and experimental values by Taneda[9], Tomboulides[10], Magnaudet[11], and Johnson[12]. According to the increment of grid points, obtained results approached the existing results. The results using Grid3 and Grid4 coincided with each other. Hereafter the results obtained by Grid4 is employed for the discussion.

Table 8.1 Properties of computational grids

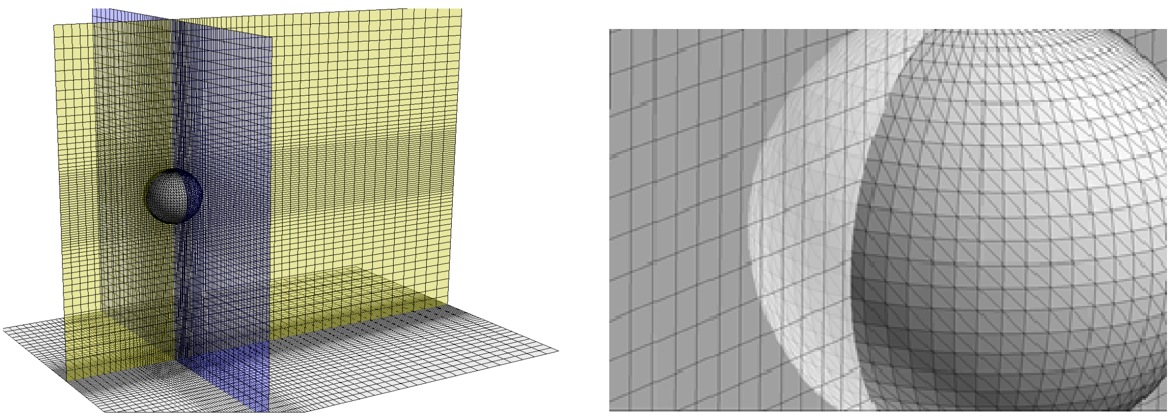
	Grid1	Grid2	Grid3	Grid4
Number of grid points	62x53x53	92x81x81	121x109x109	151x147x147
Minimum of grid space ($\Delta x = \Delta y = \Delta z$)	0.1 D	0.05 D	0.025 D	0.02 D

Table 8.2 shows the obtained drag and lift coefficients: C_D and C_L. Kim [13] and Fornberg [14] obtained C_D=1.087 and C_D=1.085 at Re=100. Our results were almost coincided to those values.

Table 8.2 Drag and lift coefficients

Re	C_D	C_L
50	1.575	1.27×10^{-2}
100	1.086	5.20×10^{-3}
150	0.879	2.23×10^{-3}

Figures 8.3 to 8.5 show 2D streamlines on xy plane and 3D streamlines behind sphere for Re= 50, 100, and 150. The length of twin vortices observed at xy plane is larger in the case of larger Reynolds number. All cases represent a doughnut shaped vortex after the sphere and the size became larger in the case of larger Reynolds number



(a) Perspective view (b) Local view

Fig. 8-1 Computational grid over a sphere

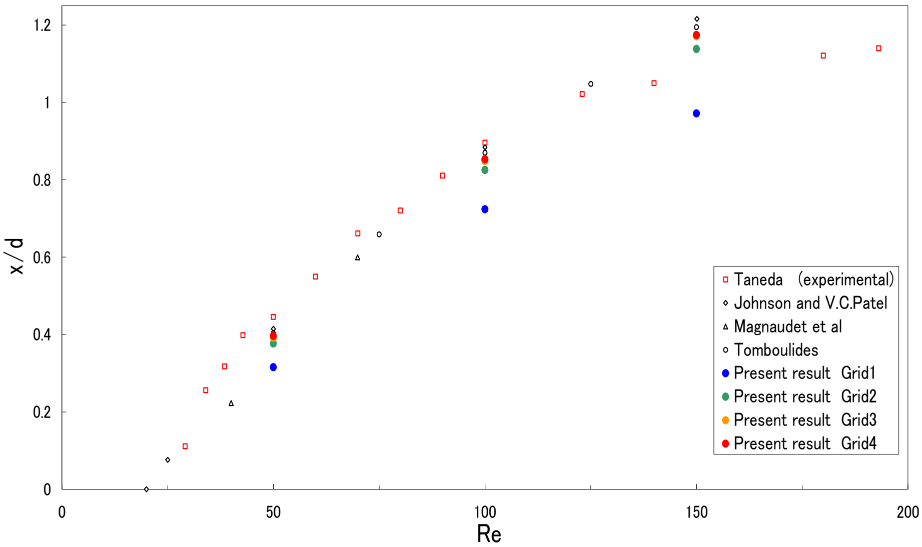
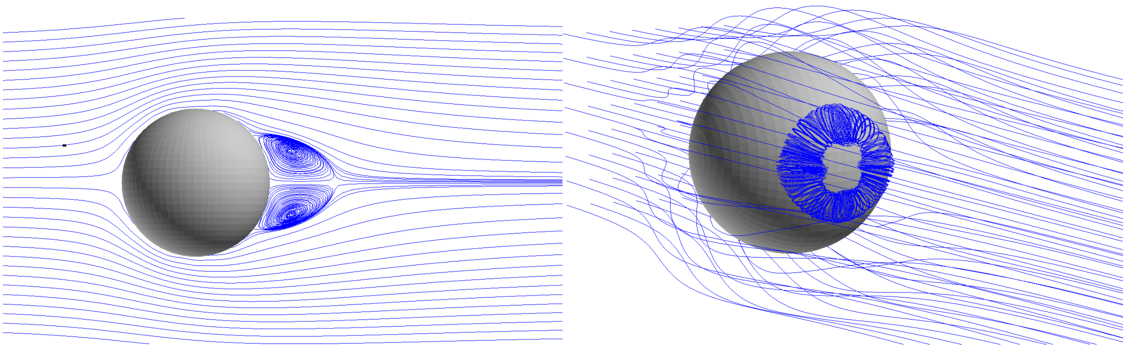
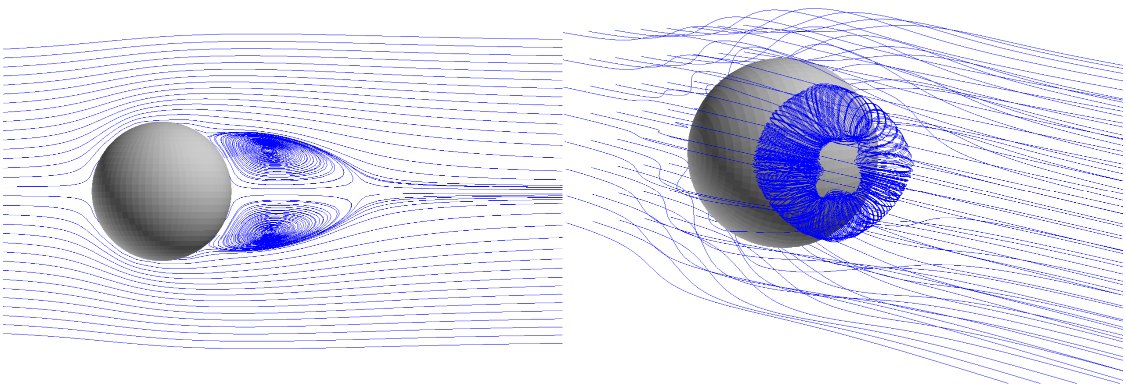


Fig. 8-2 Numerical and experimental results of vortex separation length



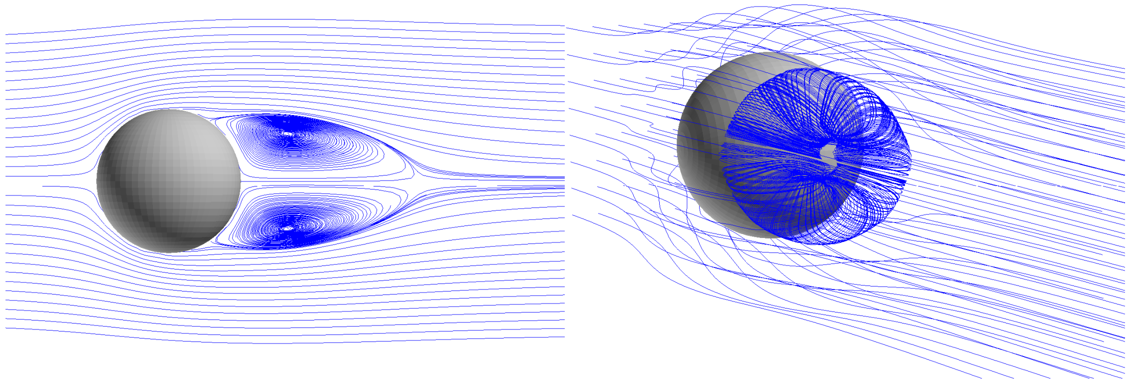
(a) 2D streamlines on *xy* plane (b) 3D streamlines behind sphere

Fig. 8-3 Calculated streamlines at Re=50



(a) 2D streamlines on *xy* plane (b) 3D streamlines behind sphere

Fig. 8-4 Calculated streamlines at Re=100



(a) 2D streamlines on xy plane (b) 3D streamlines behind sphere

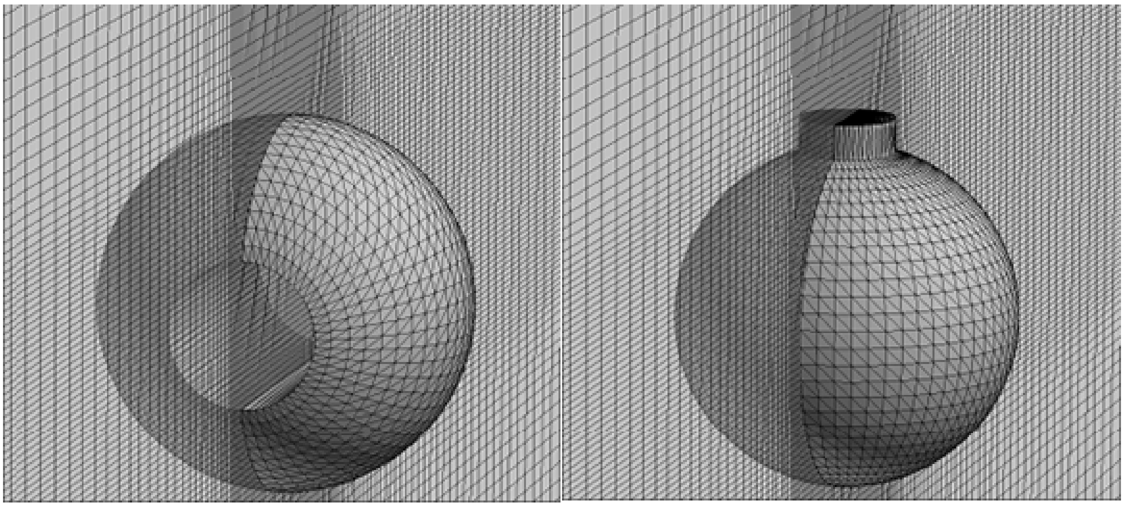
Fig. 8-5 Calculated streamlines at Re=150

The IB method is applicable to incompressible viscous flow problems over more a complicated solid body. Let me briefly introduce some results of the flows over a complex solid body where our IB method was applied.

We simulated incompressible viscous flows over deformed spheres with a tunnel hole and a cylindrical projection. The flow conditions are equivalent to those of Re=100 for the previous case.

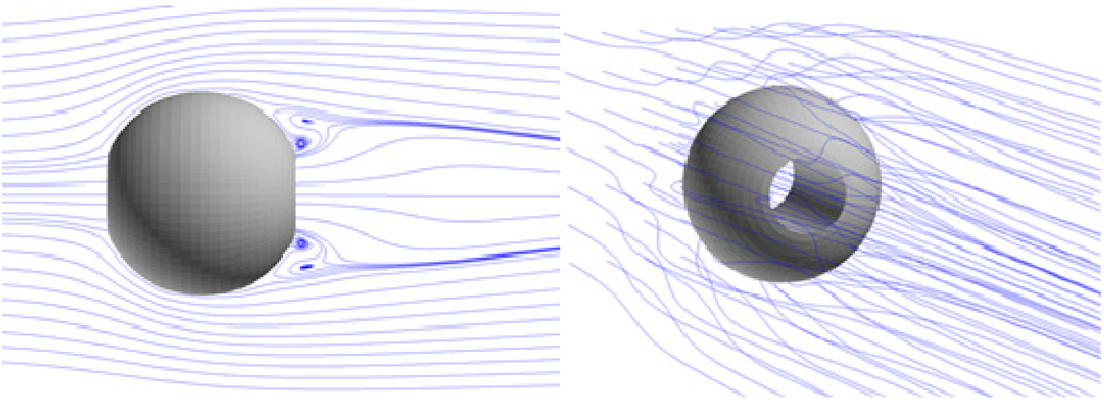
Figures 8-6 shows the surface meshes of the deformed spheres immersed in the computational grid. These surfaces are overall covered by triangle meshes. Figs. 7-7 and 7-8 show 2D streamlines on xy plane and 3D streamlines behind the deformed spheres. Vortex structures in Figs. 8-7 and 8-8 are obviously different from those in Fig. 8-4.

We further simulated a number of complex problems privately. Before the end of this lecture, some typical cases are briefly introduced. One is incompressible viscous flows over a rotating fan. Fig. 8-9 shows the fan shape and the computational grid with the immersed fan surface. Fig. 8-10 shows instantaneous u velocities on xy plane at a different time behind the rotating fan. The velocity distributions have a periodical velocity profile due to the rotation of fan. High and low velocity regions form periodically after the rotor fan. Finally, applications to two interesting problems and the obtained streamlines are introduced in Fig. 8-11. The shapes are based on free 3D frames downloaded from web pages. Even though the Reynolds number may be low, steam lines and the pressure on the body could be captured by our IB method. Note that IB methods may be applicable to lower Reynolds number flows. Further innovative ideas should be necessary for IB methods to simulate higher Reynolds number flows.



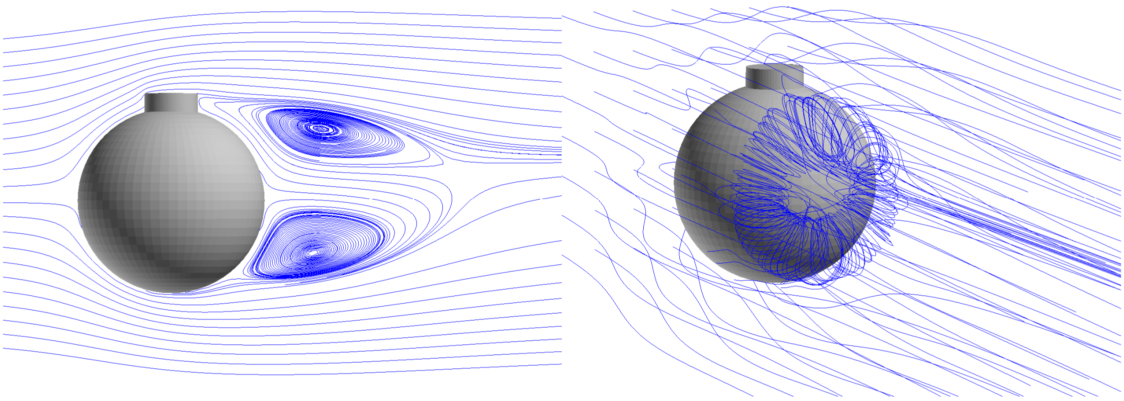
(a) With a tunnel hole (b) With a cylindrical projection

Fig. 8-6 Complex surface meshes for a sphere



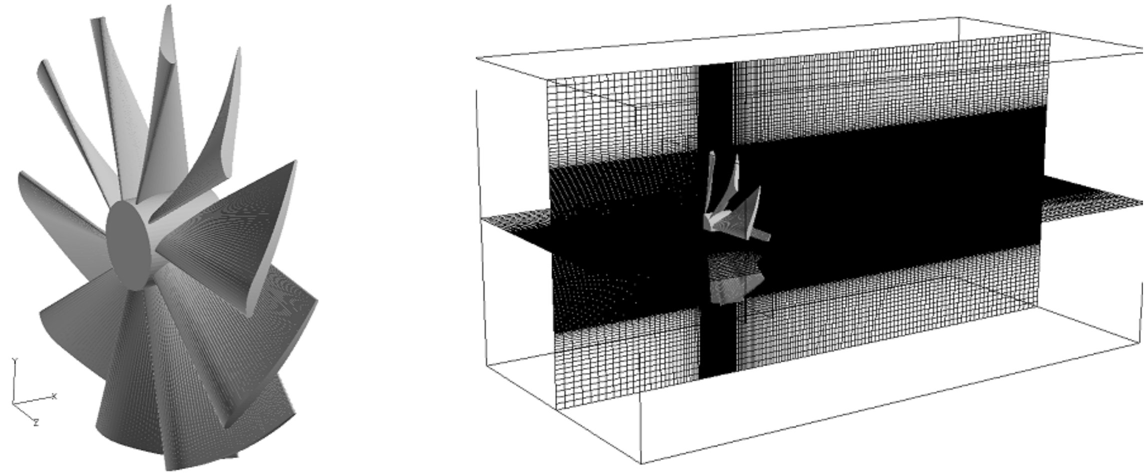
(a) 2D streamlines on xy plane (b) 3D streamlines behind sphere

Fig. 8-7 Calculated streamlines at Re=100



(a) 2D streamlines on xy plane (b) 3D streamlines behind sphere

Fig. 8-8 Calculated streamlines at Re=100



(a) Shape of rotor fan (b) Rotor fan surface immersed in the computational grid

Fig. 8-9 Rotor fan and computational grid

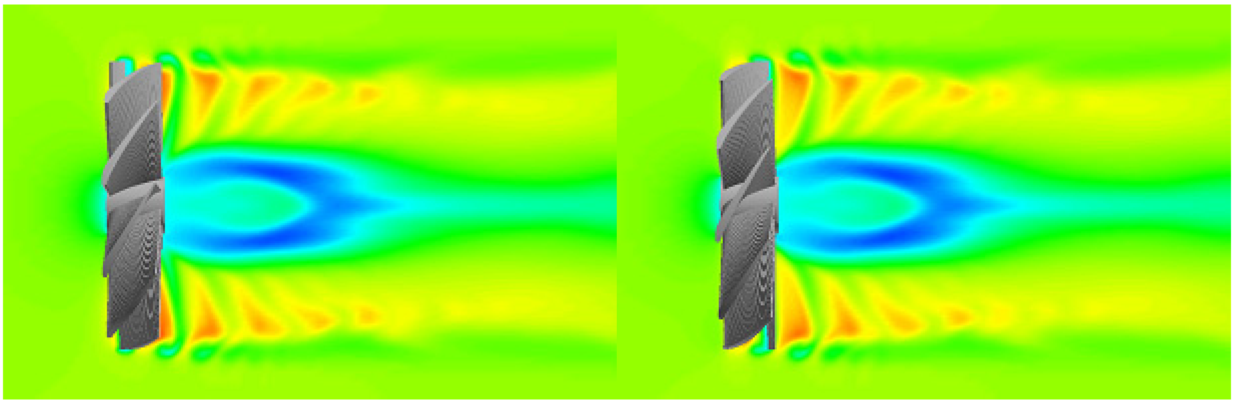
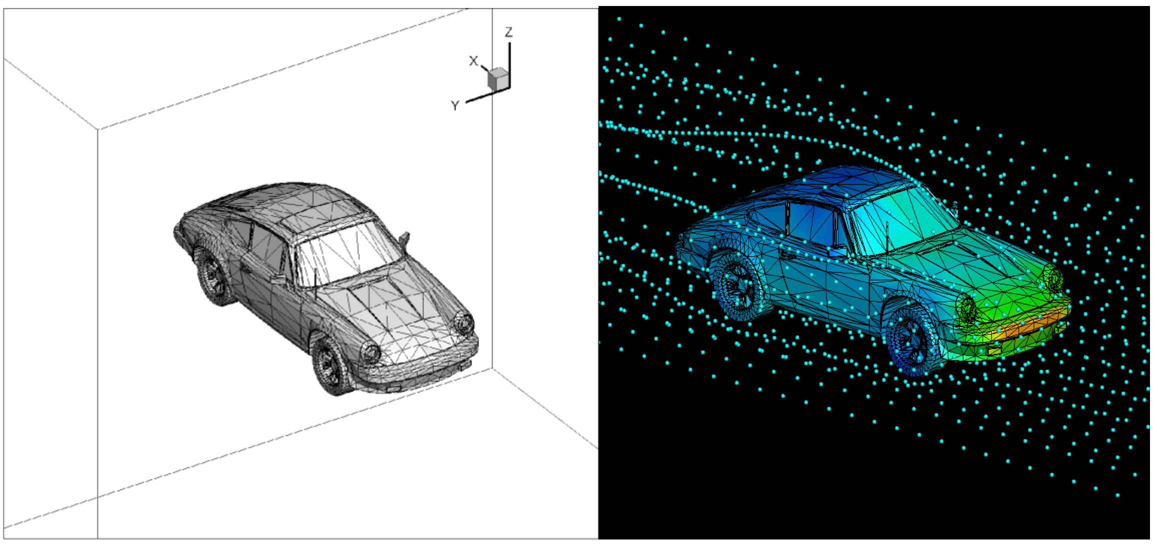


Fig. 8-10 Instantaneous u velocities on xy plane at a different time behind the rotating fan.



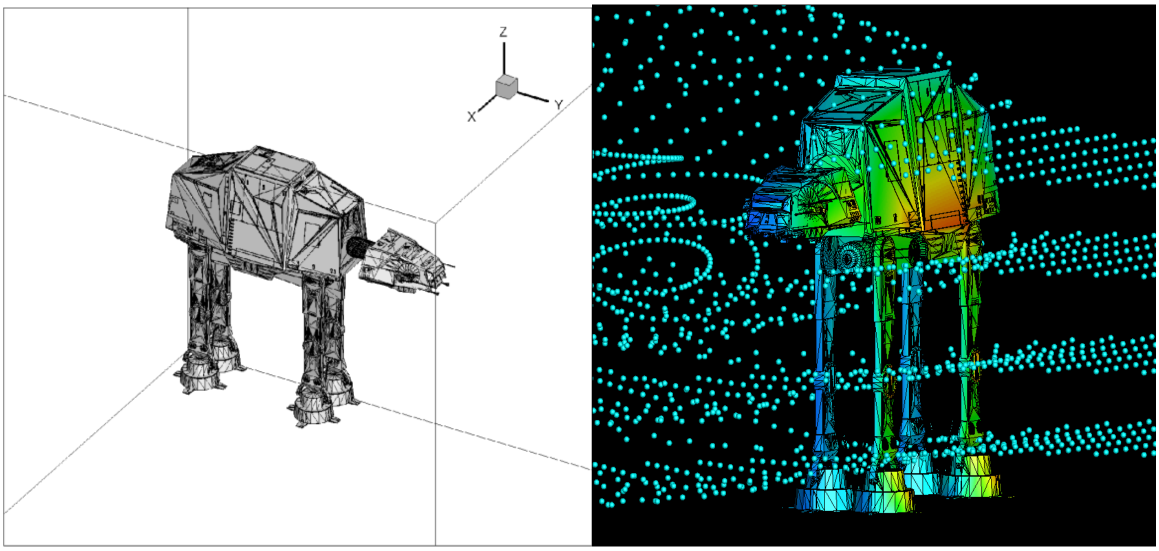


Fig. 8-11 Applications to practical 3D shapes and the obtained streamlines.

9. Concluding Remarks

This lecture note introduced three-dimensional incompressible CFD based on MAC method. MAC-based methods are still in common use for simulating incompressible viscous flows. This lecture note focused on the three-dimensional methodology to understand the basis of three-dimensional fundamental equations and the three-dimensional method. It would be helpful for readers who start CFD for simulating three-dimensional incompressible flow problems.

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